Kepler Problem under Repulsive Inverse Square Force

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Recently the author has been devoting himself to solving dust particle motion simultaneously driven by the radiation force and the gravity force. The radiation pressure is radially outward directed and varies as r^{-2}, where r is the heliocentric distance of the particle. The effective of the compound force also obeys the inverse square law. In cometary science, the strength of the radiation pressure is often defined by the following definition

$$\beta = 1 - \mu = \frac{F_{\rm rad}}{F_{\rm grav}} . \tag{1}$$

For $\beta < 1$, the dust particles are driven by attractive inverse square force, which appears to be an "effective" reduced gravity force. $\beta = 1$ corresponds to uniform linear motion as if these particles were free from any force. For $\beta > 1$, repulsive force comes into power and accelerates the particles to move afar. For the first case, traditional Keplerian orbit mechanics is sufficiently applicable to solve problem. Uniform linear motion is even simpler. For the last case, however, the author failed to find out any material that rendered solutions in a detailed manner as this scenario is frequently omitted in celestial mechanics due to the scarcity. Therefore the author showed great interest in finding the answer. It also reminisces if there exists any similar equations to Kepler's equation for determining the motion of the particles driven by such centrifugal force.

The following discussion employs polar coordinates. Let r be the radius and θ be the angular component. The origin is set at heliocenter. One shall not have much difficulty in deriving the following two differential equations from, for example, Lagrangian mechanics

$$\ddot{r} - r\dot{\theta} = \frac{k}{r^2} , \tag{2}$$

$$r^2\dot{\theta} = h , \tag{3}$$

where $k = (\beta - 1)GM_{\odot}$ is the "effective" gravitational constant, G stands for the gravitational constant and $M_{\odot}$ for the solar mass. Eliminate $\dot{\theta}$ with equation (3), and make the change of variable $u \equiv 1/r$, equation (2) can turn into the following inhomogeneous second-order linear ordinary differential equation:

$$\frac{{\rm d}^2 u}{{\rm d}\theta^2} + u = -\frac{k}{h^2} .$$

The solution to the above equation can be expressed as the following form conveniently:

$$u = \frac{k}{h^2}[e \cos(\theta - \omega) - 1] ,$$

where e and ω are constants to be fixed. We now can change back to the following form:

$$r = \frac{h^2/k}{e \cos(\theta - \omega) - 1} , \tag{4}$$

which describes a hyperbola with one of its foci at the heliocenter, however, to the contrast, the branch of the hyperbola does not orbit about the origin. We now further denote q as the perihelion distance, such that $q = r_{\min} = |a|(1 + e)$, where $\boldsymbol{a}$ is the semi-major axis of the hyperbola, a negative value. Besides, we also obtain $h = \sqrt{k|a|(e^2 - 1)}$.

Now we proceed to find r and θ as functions of time. Equations (2) and (3) lay the key to the answer. With equation (3), we yield

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{k}{r^2}\,.$$

Since $\ddot{r}$ can be expressed to the form $\ddot{r} = \dot{r}\frac{\mathrm{d}\dot{r}}{\mathrm{d}r}$, the result is therefore found to be

$$\dot{r}\,\mathrm{d}\dot{r} = \left(\frac{h^2}{r^3} + \frac{k}{r^2}\right)\mathrm{d}r\,.$$

By integrating the above equation on both sides, we obtain

$$\dot{r}^2 = -\frac{2k}{r} - \frac{h^2}{r^2} + K_1\,, \tag{5}$$

where K_1 is a constant to be fixed. A proper initial condition should help to solve the exact value. Now we apply equation (5) when the particle reaches its perihelion. In this case, $\dot{r} = 0$, consequently we manage to yield K_1 as $K_1 = k/|a|$. Hence equation (5) becomes

$$\dot{r}^2 = -\frac{2k}{r} - \frac{h^2}{r^2} + \frac{k}{|a|}\,. \tag{6}$$

Noticing that the speed of the particle, denoted as $\boldsymbol{v}$, can be expressed by relationship $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$, we can therefore change the form of equation (6) and obtain

$$v^2 = k\left(\frac{1}{|a|} - \frac{2}{r}\right). \tag{7}$$

We can address equation (7) as the vis-viva equation in repulsive hyperbolic orbit, or pseudo-vis-viva equation, as the form strikingly resembles the vis-viva equation in traditional Keplerian orbit. Simply exchanging the signs in the brackets turn equation (7) into exactly the authentic vis-via equation. The quest for r and θ as a function of time is smoothly progressing. Now, excitingly, we are approaching our target closer and closer. Equation (6) can be transformed into the following equation:

$$\mathrm{d}t = \frac{r\,\mathrm{d}r}{\sqrt{\frac{k}{|a|}r^2 - 2kr - k|a|(e^2 - 1)}}\,,$$

and further

$$\sqrt{\frac{k}{|a|^3}}\,\mathrm{d}t = \frac{r\,\mathrm{d}r}{|a|\sqrt{(r - |a|)^2 - a^2e^2}}\,.$$

We hereby introduce an auxiliary variable, denoted as F, using the equation

$$r = |a|(e\cosh F + 1)\,, \qquad (8)$$

then the differential equation above becomes

$$\sqrt{\frac{k}{|a|^3}}\,\mathrm{d}t = (e\cosh F + 1)\mathrm{d}F\,,$$

the integral of which yields

$$\sqrt{\frac{k}{|a|^3}}\,(t - T) = e\sinh F + F + K_2\,,$$

where K_2 is another integral constant to be solved with initial condition. Once again let us consider the particle at its perihelion at time T, such that the constant becomes $K_2 = 0$. Hence we obtain

$$\sqrt{\frac{k}{|a|^3}}\,(t-T) = e\sinh F + F\,, \tag{9}$$

which is the transcendental equation in variable F. Because of the striking similarity to the renowned Kepler's equation in the case of Keplerian orbit under attractive force, we may refer to equation (9) as the Kepler's equation in repulsive inverse square force or pseudo-Kepler's equation. The left side can be called the mean anomaly, denoted by M, in centrifugal hyperbolic orbit, similarly. The M can be found immediately when a specific relative time $(t-T)$ is given, after which equation (9) must be solved for the auxiliary variable F. Likewise, there does not exist any straightforward algorithm to solve F unfortunately, however, it can be solved iteratively by proper initial guess by means of Newton-Raphson method. Once F is yielded, the heliocentric distance r can be obtained as well, and therefore, along with equation (4), the angular component θ will be solved. But in this article, it is intriguing to also express θ as a function of F.

For simplicity, we set $f = \theta - \omega$. With equation (8), equation (4) becomes

$$\cos f = \frac{e + \cosh F}{1 + e\cosh F}\,.$$

Applying half-angle formulae for sine and cosine in trigonometrics respectively, i.e. $2\,\sin^2 f = 1 - \cos f$, and $2\cos^2 f = 1 + \cos f$, one will find it easy to derive the following two equations

$$\sin\frac{f}{2} = \sqrt{\frac{(e-1)\sinh^2\frac{F}{2}}{1 + e\cosh F}}\,,$$
$$\cos\frac{f}{2} = \sqrt{\frac{(e+1)\cosh^2\frac{F}{2}}{1 + e\cosh F}}\,.$$

The upper equation is divided by the lower, whereby it is found that

$$\tan\frac{f}{2} = \sqrt{\frac{e-1}{e+1}}\tanh\frac{F}{2}\,. \tag{10}$$

Heretofore we succeed to solve the problem perfectly. We can conclude that particles driven by repulsive inverse square force follows similarly the equations applied to those moving along their individual Keplerian orbits. We do not have to make numerous changes to modify equations applicable under attractive hyperbolic scenarios, but flipping the signs at several places in the equations works sufficiently.

REFERENCE

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