Demonstration of the Lorentz Transformation

Hereby I will describe my own way to demonstrate the Lorentz Transformation.

Consider that we now have two Cartesian frames, $S(x, y, z, t)$ and $S'(x', y', z', t')$, in which S' moves in the x-direction of S with a uniform velocity v. Despite the motion, all of the three corresponding axes of S and S' remain parallel. We further assume that at time $t = 0$, the two reference frames completely overlapped.

Under the Lorentz Transformation, the speed of light is an invariant between different inertial reference frames. We hereby set off with this rule in mind.

Suppose that a spherical wave of light is emitted at time $t = 0$. We should have the following two equations

$$x^2 + y^2 + z^2 = c^2t^2 \tag{1}$$

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \tag{2}$$

There is no relative motion within the two-dimensional space of $O - yz$ and $O' - y'z'$. In other words, these two spaces remain identical to each other. Hereby I venture to set the following relationships:

$$x' = f(x, t) \tag{3}$$

$$y' = y$$

$$z' = z$$

$$t' = g(x, t) \tag{4}$$

On the other hand, we have another two differential equations:

$$dx^2 + dy^2 + dz^2 = c^2dt^2 \tag{5}$$

$$dx'^2 + dy'^2 + dz'^2 = c^2dt'^2 \tag{6}$$

From mathematics, we obtain

$$dx' = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt \tag{7}$$

$$dt' = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial t}dt \tag{8}$$

Now take (7) and (8) into the differential equation (6):

$$\left[\left(\frac{\partial f}{\partial x}\right)^2 - c^2\left(\frac{\partial g}{\partial x}\right)^2\right]dx^2 + dy^2 + dz^2 + 2\left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial t} - c^2\frac{\partial g}{\partial x}\frac{\partial g}{\partial t}\right)dxdt = \left[c^2\left(\frac{\partial g}{\partial t}\right)^2 - \left(\frac{\partial f}{\partial t}\right)^2\right]dt^2$$

By comparison with (5), we yield

$$\begin{cases} \left(\frac{\partial f}{\partial x}\right)^2 - c^2\left(\frac{\partial g}{\partial x}\right)^2 = 1 \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial t} - c^2\frac{\partial g}{\partial x}\frac{\partial g}{\partial t} = 0 \\ c^2\left(\frac{\partial g}{\partial t}\right)^2 - \left(\frac{\partial f}{\partial t}\right)^2 = c^2 \end{cases} \tag{9}$$

Now we differentiate (1) and (2) simultaneously:

$$xdx + ydy + zdz = c^2tdt \tag{10}$$

$$x'dx' + y'dy' + z'dz' = c^2t'dt' \tag{11}$$

Again we substitute x' and t' in (10) with (7) and (8), later is subtracted by (9), we yield:

$$\left(x - f\frac{\partial f}{\partial x} + c^2g\frac{\partial g}{\partial x}\right)dx = \left(f\frac{\partial f}{\partial t} + c^2t - c^2g\frac{\partial g}{\partial t}\right)dt$$

Since x is independent from t, immediately we obtain

$$\begin{cases} x - f\frac{\partial f}{\partial x} + c^2 g\frac{\partial g}{\partial x} = 0 \\ f\frac{\partial f}{\partial t} + c^2 t - c^2 g\frac{\partial g}{\partial t} = 0 \end{cases} \tag{12}$$

We partially differentiate these two relationships with x and t respectively. Notice that we have (9). Hence we yield:

$$\begin{cases} f\frac{\partial^2 f}{\partial x^2} = c^2 g\frac{\partial^2 g}{\partial x^2} \\ f\frac{\partial^2 f}{\partial t^2} = c^2 g\frac{\partial^2 g}{\partial t^2} \end{cases} \tag{13}$$

It will not be difficult to notice that the two relationships in (13) in fact are identical to each other. In other words, they appear symmetric. Now we proceed to partially differentiate the first equation with x and the third one with t in (9) respectively. We therefore obtain

$$\begin{cases} \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2} = c^2\frac{\partial g}{\partial x}\frac{\partial^2 g}{\partial x^2} \\ \frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial g}{\partial t}\frac{\partial^2 g}{\partial t^2} \end{cases} \tag{14}$$

Now we assume $\frac{\partial^2 f}{\partial q^2} \neq 0$ and $\frac{\partial^2 g}{\partial q^2} \neq 0$, where q stands for x or t. Therefore, from (13) and (14) we can yield the following two relationships without any difficulty

$$\begin{cases} g\frac{\partial f}{\partial x} = f\frac{\partial g}{\partial x} \\ g\frac{\partial f}{\partial t} = f\frac{\partial g}{\partial t} \end{cases} \tag{15}$$

Each side of the first equation and of the second equation is simultaneously multiplied by dx and dt, respectively, which later is summed together. We obtain

$$g\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt\right) = f\left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial t}dt\right)$$

or,

$$gdf = fdg \tag{16}$$

From (16) we obtain that f is proportional to g. The ratio, which is a non-zero constant, is denoted by k, namely, $f = kg$ If the assumption, $\frac{\partial^2 f}{\partial q^2} \neq 0$ and $\frac{\partial^2 g}{\partial q^2} \neq 0$ is correct, we should be able to solve k perfectly. However, this is unfortunately not the case, as we apply the relationship of $f = kg$ into (9), and find that there are more than on errors with the assumption:

$$\begin{cases} \left(\frac{\partial g}{\partial t}\right)^2 = \frac{1}{k^2 - c^2} \\ \left(\frac{\partial g}{\partial t}\right)^2 = \frac{c^2}{c^2 - k^2} \end{cases} \tag{17}$$

However the ratio k should be, $\frac{\partial g}{\partial t}$ is a constant, independent from variants x and t. Under such circumstance, $\frac{\partial^2 g}{\partial t^2} = 0$ stands clearly, which is evidently contradictory to the assumption. Therefore, the assumption is flawed. By completely the same manner, instead, we must have $\frac{\partial^2 f}{\partial q^2} = 0$ and $\frac{\partial^2 g}{\partial q^2} = 0$, where q stands for both x and t. Now that $\frac{\partial f}{\partial q} \neq 0$ and $\frac{\partial g}{\partial q} \neq 0$ are valid, consequently, both of the functions $x' = f(x,t)$ and $t' = g(x,t)$ are linear. Things become much easier. We assume four coefficients for x' and t':

$$\begin{cases} x' = \alpha x + \beta t \\ t' = \gamma x + \delta t \end{cases} \tag{18}$$

According to (9), we easily yield

$$\begin{cases} \alpha^2 - c^2\gamma^2 = 1 \\ \alpha\beta = c^2\gamma\delta \\ c^2\delta^2 - \beta^2 = c^2 \end{cases} \tag{19}$$

Notice that $x = vt$ actually describes the motion of the origin point of the reference frame S'. Hence we ought to have another relationship $0 = \alpha vt + \beta t$, namely, $\alpha v = -\beta$. With the four equations, the four coefficients can be solved as follows

$$\begin{cases} \alpha = \dfrac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \\ \beta = -\dfrac{v}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \\ \gamma = -\dfrac{\frac{v}{c^2}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \\ \delta = \dfrac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \end{cases}$$

Finally, the Lorentz Transformation between coordinates in S and S' is given by

$$\begin{cases} x' = \dfrac{x - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - \frac{v}{c^2}x}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \end{cases}$$

We manage to observe that the Galilean Transformation is in fact an approximation in condition that $\frac{v}{c} \ll 1$ is valid. For circumstances in which $\frac{v}{c} \ll 1$ does not stand, the Lorentz Transformation should be applied.