Orbit Mechanics for Dust Particles of Comets

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Abstract

This article discusses about orbit mechanics of dust particles released by its parent comet, so as to determine their trajectories. Only kinetic concepts are within the consideration for simplicity, and problems in fluid dynamics are omitted. This article assumes that the dust particles released from the nucleus are driven only by solar gravity and pressure of the solar radiation simultaneously, without any intervention from the drag force which is attributed to the expanding gas in the inner nuclear region. Also is that this article pressumes that the motion of the dust particles is free from any collisions and the released velocity is entirely identical to that of the comet nucleus. The Appendix A of paper A Theory of Dust Comets. I. Model and Equations written by Michael Finson and Ronald Probstein has expounded the methodolgy to determine the position of the released dust particles, however, the author of this article was not contented with the discussion as only comets orbiting in parabolic orbits would satisfy the requirements for application of the equations, and these members are only a portion of the great swarms. Therefore, the author wish to quest for a general type to determine the positions of the dust particles, in particular, with respect to the comet nucleus.

We consider the observation time at t_c ($t_c = 0$ at the comet's perihelion, $t_c < 0$ for pre-perihelion and $t_c > 0$ for post-perihelion respectively). A dust particle is released from the comet nucleus τ earlier, the motion of which is driven by solar gravity and pressure of solar radiation simultaneously. In cometary science, since the strenght of pressure of radiation force varies as r_d^{-2}, where r_d is the heliocentric distance of the dust particle, whereby it obeys the same law as that of gravity, it is frequently defined by the following form

$$\beta = 1 - \mu = \frac{F_{\text{rad}}}{F_{\text{grav}}}\,. \tag{1}$$

β can be categorised into three groups, i.e. $0 < \beta < 1$, $\beta = 1$ and $\beta > 1$. The first case corresponds to a lower effective gravity field, the second corresponds to a zero-force field and the last one corresponds to a repulsive inverse square force field. Unfortunately, the paper by Finson and Probstein only discussed motions subject to $\beta < 1$, so it therefore shows even more necessity to review and render the whole discussions in a more complete manner.

In order to conveniently describe the comet tails, the cometocentric coordinates ξ, η are frequently put into utilization, as shown in Figure 1. Both of the axes are in the comet's orbital plane, with ξ positive directed outward from the Sun radially, and η directed tangentially to ξ, positive opposite to the comet's motion along the orbit.

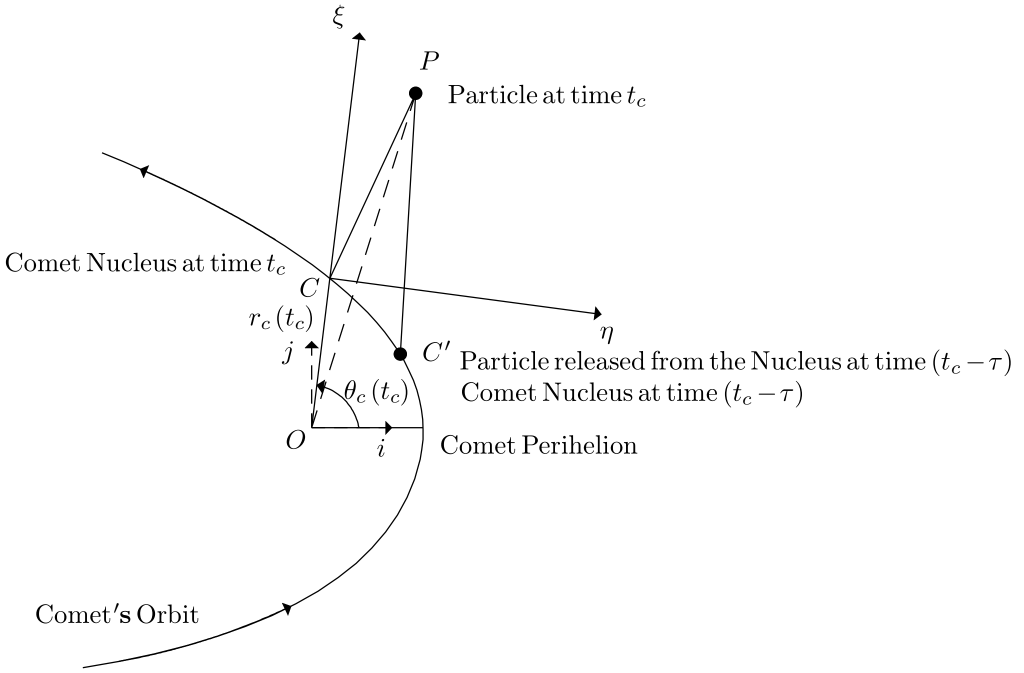

Figure 1.

Nevertheless, we need to determine the comet nucleus in the first step. The nuclues iteself admittedly is also influenced by solar radiation pressure, however, the effect is trivial enough within a short spell of observation period. So only gravitation drag will play the role. Let r_c be the heliocentric distance of the comet nucleus, θ_c the angular component in the polar coordinates ($\theta_c = 0$ at perihelion), q_c the perihelion distance, e_c the eccentricity of the comet's orbit, $M_\odot$ the solar mass, and G the gravitational constant. By playing with fundamental physics, or directly from knowledge in our celestial mechanics textbooks, we will know that the orbit of the comet nucleus is given by

$$r_c(t) = \frac{q_c(1+e)}{1+e_c\cos\theta_c(t)}\,. \tag{2}$$

For $0 \leqslant e_c < 1$, equation (2) describes an ellipse with the origin at one of its foci, $e_c = 1$ a parabola with the origin at its focus, and $e_c > 1$ is a branch of hyperbola enclosing the origin at its corresponding focus.

Once the observation time t is specified, Kepler's equation is applicable for solving $r_c(t)$ and $\theta_c(t)$. Since there are three types of orbits in total, we have three different transcendental equations correspondingly. For elliptical orbits, the equation is

$$\sqrt{GM_\odot\left(\frac{1-e_c}{q_c}\right)^3}\,t = E_c - e_c\sin E_c\,, \tag{3.1}$$

for parabolic, the equation becomes

$$\sqrt{\frac{GM_\odot}{2\,q_c^3}}\,t = D_c + \frac{1}{3}D_c^3\,, \tag{3.2}$$

and finally, for hyperbolic, closely linked to that of elliptical, the equation goes as

$$\sqrt{GM_\odot\left(\frac{e_c-1}{q_c}\right)^3}\,t = e_c\sinh F_c - F_c\,, \tag{3.3}$$

where E_c, D_c and F_c are all auxiliary time-dependence variables to be solved respectively for elliptical, parabolic and hyperbolic orbits. Thanks for advanced modern technology, researchers no longer feel too much difficulty in solving the transcendental equations (3.1) and (3.3) for E_c and F_c respectively in fast iterative method. The left sides of the three equations are all termed Mean Anomaly, often denoted by letter M in celestial mechanics. The linkage between the auxiliary variables and the components in polar coordinates is as follows. For elliptical,

$$r_c(t) = \frac{q_c}{1-e_c}(1-e_c \cos E_c)\,, \tag{4.1a}$$

$$\tan\frac{\theta_c(t)}{2} = \sqrt{\frac{1+e_c}{1-e_c}}\tan\frac{E_c}{2}\,, \tag{4.1b}$$

for parabolic,

$$r_c(t) = q_c(1+D_c^2)\,, \tag{4.2a}$$

$$\tan\frac{\theta_c}{2} = D_c\,, \tag{4.2b}$$

and at last, for hyperbolic orbit,

$$r_c(t) = \frac{q_c}{e_c-1}(e_c \cosh F_c - 1)\,, \tag{4.3a}$$

$$\tan\frac{\theta_c}{2} = \sqrt{\frac{e_c+1}{e_c-1}}\tanh\frac{F_c}{2}\,. \tag{4.3b}$$

Anyway the position of the comet nucleus can now be determined at any given time. Now it is time to determine the orbit of the dust particles in a similar but somewhat complicated manner. We assume that the dust particle is emitted from the nucleus with initial zero speed with respect to the nucleus. The initial conditions of the particles will be fixed at the time of release $t_c - \tau$. The compound force $\vec{\mathrm{F}}$ that the particle is subject to is

$$\vec{\mathrm{F}} = -\frac{\mu G M_\odot}{r_d^2}\vec{r}_0\,,$$

where $\vec{r}_0$ is the unit radial vector, directed outward from the sun. There exist three categories of β value, namely, $0 < \mu < 1$, $\mu = 0$, and $\mu < 0$. We are now discussing the first case, where the particle is driven by attractive reduced gravity, in the same methodology as that in Appendix A of the paper by Finson and Probstein.

We need four constants to determine the trajectory of the particle. The first pair determines the semimajor axis a_d and the eccentricity e_d of the orbit, whereas another pair of parameters gives the orientation of its perihelion with respect to that of the comet's orbit, α_d, and the time of perihelion, t_{0d}. In case of $e_d = 1$, i.e. the orbit is a parabola, we introduce yet another parameter called perihelion distance of the dust particle, denoted as q_d so as to avoid an infinite a_d of the orbit. The relationship between a_d, e_d and q_d is

$$a_d = \frac{q_d}{1-e_d}\,. \tag{5}$$

Since the effective force is attractive and obeys the inverse square law, concepts in Keplerian orbits are still applicable. The orbit path is

$$r_d(t) = \frac{a_d\,(1-e_d^2)}{1+e_d \cos\left[\theta_d\,(t) - \alpha_d\right]}\,. \tag{$A1$}$$

It requires that at the time of release the dust particle and the nucleus must coincides, such that

$$r_d(t_c-\tau) = r_c(t_c-\tau)\,, \theta_d(t_c-\tau) = \theta_c(t_c-\tau)\,. \tag{6}$$

Also is that at emission the velocity of the dust particle must equal to that of the comet nucleus, therefore, the respective radial and tangential components must be equivalent to each other. Henceforth we have

$$\dot{r}_c(t_c-\tau)=\dot{r}_d\,(t_c-\tau)\,,\dot{\theta}_c\,(t_c-\tau)=\dot{\theta}_d\,(t_c-\tau)\,. \tag{7}$$

It is not difficult to yield the specific forms for equations (6) and (7) with equations (A1) and (2). We have

$$\dot{r}=\frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta}\,, \tag{8}$$

and the law of angular momentum conservation

$$r^2\dot{\theta}=h\,, \tag{9}$$

where h is a constant. In Keplerian orbit, it shows that $h_c = \sqrt{GM_\odot\,q_c\,(1+e_c)}$, and $h_d=\sqrt{\mu GM_\odot\,q_d\,(1+e_d)}$. Thereby we are able to yield the following equations

$$\dot{r}_c(t_c-\tau)=\sqrt{\frac{GM_\odot}{q_c\,(1+e_c)}}\,e_c\sin\theta_c\,(t_c-\tau)\,,\dot{r}_d\,(t_c-\tau)=\sqrt{\frac{\mu GM_\odot}{q_d\,(1+e_d)}}\,e_d\sin\left[\theta_d\,(t_c-\tau)-\alpha_d\right], \tag{A2}$$

and for the angular components,

$$\dot{\theta}_c\,(t_c-\tau)=\frac{\sqrt{GM_\odot\,q_c\,(1+e_c)}}{r_c^2\,(t_c-\tau)}\,,\dot{\theta}_d\,(t_c-\tau)=\frac{\sqrt{\mu GM_\odot\,q_d\,(1+e_d)}}{r_d^2\,(t_c-\tau)}\,. \tag{A3}$$

With equations (A2) and (A3) we manage to reveal the following two relationships

$$q_c\,(1+e_c)=\mu a_d\,(1-e_d^2)=\mu q_d\,(1+e_d)\,, \tag{A4}$$

$$e_c\sin\theta_c\,(t_c-\tau)=\mu e_d\sin\left[\theta_c\,(t_c-\tau)-\alpha_d\right]. \tag{A5}$$

Had we solved parameter e_d, α_d could be solved as well

$$\alpha_d=\theta_c\,(t_c-\tau)-\arcsin\left[\frac{e_c}{\mu e_d}\sin\theta_c\,(t_c-\tau)\right]. \tag{A6}$$

So we are forced to find out e_d first, and it is actually not very difficult. Bear in mind that we have a relationship given by the first equation in (6), and combine with (A1) we manage to yield

$$\mu e_d\cos\left[\theta_c\,(t_c-\tau)-\alpha_d\right]=\beta+e_c\cos\theta_c\,(t_c-\tau)\,.$$

Using the relationship $\cos^2\theta+\sin^2\theta=1$, we thus succeed to solve e_d as

$$e_d=\sqrt{1-\frac{1-e_c^2}{\mu^2}+\frac{2\beta q_c\,(1+e_c)}{\mu^2\,r_c\,(t_c-\tau)}}\,. \tag{A7}$$

Along with equation (A4) the semimajor axis of the particle's orbit a_d can now be solved too:

$$a_d=\frac{\mu q_c\,r_c\,(t_c-\tau)}{(1-e_c)\,r_c\,(t_c-\tau)-2\beta q_c}\,. \tag{A8}$$

In case of $e_d = 1$, we choose to use q_d instead and yield

$$q_d = a_d\,(1 - e_d) = \frac{q_c\,(1 + e_c)}{2\,\mu}\,. \tag{A9}$$

Hitherto we manage to obtain three of the four parameters. Before proceeding further, we need to take a careful look back at equation (A6). Correct quadrant should be taken into account before a correct α_d is found. It is worthwhile to point out that Finson and Probstein's paper appears to make a mistake. Here comes our rectification. If $\mu > q_c\,(1+e_c)/r_c\,(t_c - \tau)$, the second term must be an angle in the second or third quadrant. If $\mu = q_c\,(1+e_c)/r_c\,(t_c - \tau)$, the angle equals to $\pm 90°$. Noticing that we are currently discussing about $0 < \mu < 1$, we can conclude that this would never happen unless the release time $(t_c - \tau)$ is far from the perihelion time. We cannot determine which quadrant of the angle will be if $\mu < q_c\,(1+e_c)/r_c\,(t_c - \tau)$.

The rest of the task to solve the last parameter t_{0d}, and we have to classify e_d here into three groups, i.e. $e_d < 1$, $e_d = 1$, and $e_d > 1$, respectively for elliptical, parabolic and hyperbolic orbit, in that the difference comes from the corresponding distinguishing Kepler's equations. If the comet itself is orbiting in parabola or hyperbola, the potential energy of the dust particle is not sufficient to trap it and therefore the trajectory must be a hyperbola, i.e. $e_d > 1$. However for comets in low-eccentricity orbits, all of the three types are possible. Nevertheless, let us consider an elliptical orbit first.

In this case, the Kepler's equation becomes

$$E(t) - e_d \sin E(t) = \sqrt{\frac{\mu G M_\odot}{a_d^3}}\,(t - t_{0d})\,. \tag{A10.a}$$

$E(t)$ could be solved from the following equation applied for elliptical Keplerian orbit

$$E(t) = E(t_c - \tau) = 2\arctan\left[\sqrt{\frac{1-e_d}{1+e_d}}\,\tan\frac{\theta_c\,(t_c - \tau) - \alpha_d}{2}\right]. \tag{A10.b}$$

Henceforth, we yield

$$t_{0d} = t_c - \tau - \sqrt{\frac{a_d^3}{\mu G M_\odot}}\,[E(t_c - \tau) - e_d \sin E(t_c - \tau)]\,. \tag{A10.c}$$

For parabolic orbits, Barker's equation is applicable

$$\frac{1}{3}\,D^3(t) + D(t) = \sqrt{\frac{\mu G M_\odot}{2\,q_d^3}}\,(t - t_{0d})\,, \tag{A11.a}$$

in which $D(t)$ can be solved immediately from the following equation

$$D(t_c - \tau) = \tan\frac{\theta_d\,(t_c - \tau) - \alpha_d}{2}\,. \tag{A11.b}$$

It comes that

$$t_{0d} = t_c - \tau - \sqrt{\frac{2\,q_d^3}{\mu G M_\odot}}\left[\frac{1}{3}\,D^3(t_c - \tau) + D(t_c - \tau)\right] \tag{A11.c}$$

for particles moving along parabolic orbits. Finally, for hyperbolic case, we have

$$e_d \sinh F(t) - F(t) = \sqrt{-\frac{\mu G M_\odot}{a_d^3}}\,(t - t_{0d})\,, \tag{A12.a}$$

where $F(t)$ could be solved from

$$F(t_c-\tau)=2\,\mathrm{arctanh}\left[\sqrt{\frac{e_d-1}{e_d+1}}\tan\frac{\theta_c\,(t_c-\tau)-\alpha_d}{2}\right], \tag{A12.b}$$

and therefore, we obtain

$$t_{0d}=t_c-\tau-\sqrt{-\frac{a_d^3}{\mu GM_\odot}}\,[e_d\sinh F(t_c-\tau)-F(t_c-\tau)]\,. \tag{A12.c}$$

Hereby we manage to solve all of the four parameters and therefore complete all the discussions for the scenario for $0<\mu<1$, i.e. $\beta<1$. In literally the same methodology, we can now further progress to discuss $\mu<0$, viz., $\beta>1$, where the dust particles are subject to repulsive inverse square force.

The author's article *Kepler Problem under Repulsive Inverse Square Force* has already given the orbits that these particles will move along. So we are not going to repeat the procedures but exploit the results directly. The parameters determining the particle's orbit remain the same, namely, e_d, a_d, α_d and t_{0d}. The repulsive hyperbolic orbit is defined by the following equation

$$r_d(t)=\frac{a_d\,(1-e_d^2)}{e_d\cos\left[\theta_d\,(t)-\alpha_d\right]-1}\,. \tag{B1}$$

The Kepler's equation under the repulsive force becomes

$$e_d\sinh F(t)+F(t)=\sqrt{\frac{\mu GM_\odot}{a_d^3}}\,(t-t_{0d})\,. \tag{B2}$$

The following equation will help to solve $F(t)$ if two of the four parameters, e_d and α_d are known:

$$\sqrt{\frac{e_d-1}{e_d+1}}\tanh\frac{F\,(t)}{2}=\tan\frac{\theta_d\,(t)-\alpha_d}{2}\,. \tag{B3}$$

The initial conditions of the dust particle at emission remain unchanged as defined by equations (6) and (7). By exactly the same means as that applied to the case of $0<\mu<1$, we are able to solve the four parameters, through the following equations

$$e_d=\sqrt{1-\frac{1-e_c^2}{\mu^2}+\frac{2\,\beta q_c\,(1+e_c)}{\mu^2\,r_c\,(t_c-\tau)}}\,, \tag{B4}$$

$$a_d=-\frac{\mu q_c\,r_c\,(t_c-\tau)}{(1-e_c)\,r_c\,(t_c-\tau)-2\,\beta q_c}\,, \tag{B5}$$

$$\alpha_d=\theta_c\,(t_c-\tau)-\arcsin\left[-\frac{e_c}{\mu e_d}\sin\theta_c\,(t_c-\tau)\right], \tag{B6}$$

$$t_{0d}=t_c-\tau-\sqrt{\frac{a_d^3}{\mu GM_\odot}}\,[e_d\sinh F(t_c-\tau)+F(t_c-\tau)]\,, \tag{B7}$$

from which we learn that equation (B4) is entirely the same as equation (A7) in the case of attractive force, and that a_d and α_d would share unified forms by the following modification

$$a_d=\frac{|\mu|\,q_c\,r_c\,(t_c-\tau)}{(1-e_c)\,r_c\,(t_c-\tau)-2\,\beta q_c}\,, \tag{10}$$

$$\alpha_d=\theta_c\,(t_c-\tau)-\arcsin\left[\frac{e_c}{|\mu|e_d}\sin\theta_c\,(t_c-\tau)\right]. \tag{11}$$

Now we almost complete the discussion for dust particles moving along accelerated trajectories. Yet before we step onto addressing the last case of $\mu=0$, where the dust particles are moving in different uniform linear trails dependent upon the release time, we need to transform the polar coordinates r_d, θ_d into the desired cometocentric coordinates η, ξ, as we plan to employ a direct way to yield η, ξ without much dealing with the heliocentric polar coordinate r_d, θ_d any more in case of $\mu=0$. From Figure 1 we can easily find the following conversion relationships to the cometocentric system:

$$\eta\left(t\right)=r_d\left(t\right)\sin\left[\theta_c\left(t\right)-\theta_d\left(t\right)\right],\xi\left(t\right)=r_d\left(t\right)\cos\left[\theta_c\left(t\right)-\theta_d\left(t\right)\right]-r_c\left(t\right). \tag{12}$$

Finally, here comes the discussion for $\mu=0$. As the dust particle is not subject to any force, its velocity remains always the same along its motion, namely, its initial velocity at emission time $(t_c-\tau)$. Furthermore, our assumption is that the particle is released from the nucleus with no relative velocity to the comet nucleus, wherefore this initial velocity is equal to the velocity of the comet nucleus at time $(t_c-\tau)$. For convenience vector calculation is applied. From Figure 1 we learn that the conversion towards the cometocentric coordinates can be achieved through the following method

$$\eta=\overrightarrow{CP}\cdot\vec{r}_0\left(t_c\right),\xi=\overrightarrow{CP}\cdot\vec{\theta}_0(t_c), \tag{$C1$}$$

and

$$\overrightarrow{CP}=\vec{r}_c(t_c-\tau)+\overrightarrow{C'P}-\vec{r}_c(t_c), \tag{$C2$}$$

where $\overrightarrow{C'P}$, $\vec{r}_c\left(t_c-\tau\right)$, and $\vec{r}_c\left(t_c\right)$ can be expressed by the following equations

$$\overrightarrow{C'P}=\vec{v}_d\cdot\tau, \tag{$C3.1$}$$
$$\vec{r}_c(t_c-\tau)=r_c\left(t_c-\tau\right)\left[\cos\theta_c\left(t_c-\tau\right)\vec{i}+\sin\theta_c\left(t_c-\tau\right)\vec{j}\,\right], \tag{$C3.2$}$$
$$\vec{r}_c(t_c)=r_c\left(t_c\right)\left[\cos\theta_c\left(t_c\right)\vec{i}+\sin\theta_c\left(t_c\right)\vec{j}\,\right]. \tag{$C3.3$}$$

Both $\vec{i}$ and $\vec{j}$ are unit vectors in the Cartesian coordinates, and $\vec{v}_d$ is the velocity of the dust particle at point C', which can be shown as the following equations in heliocentric polar coordinates system:

$$\vec{v}_d=\dot{r}_c\left(t_c-\tau\right)\vec{r}_0\left(t_c-\tau\right)+r_c\left(t_c-\tau\right)\dot{\theta}_c\left(t_c-\tau\right)\vec{\theta}_0\left(t_c-\tau\right),$$

in which we have the following conversion relationships between the unit vecotrs respectively in the Cartesian coordinates and the polar coordinates

$$\vec{r}_0\left(t_c-\tau\right)=\cos\theta_c\left(t_c-\tau\right)\vec{i}+\sin\theta_c\left(t_c-\tau\right)\vec{j},$$
$$\vec{\theta}_0\left(t_c-\tau\right)=-\sin\theta_c\left(t_c-\tau\right)\vec{i}+\cos\theta_c\left(t_c-\tau\right)\vec{j}.$$

Remembering that we have equations (8) and (9), we manage to express parameter $\vec{v}_d$ in the Cartesian coordinates as

$$\vec{v}_d=\sqrt{\frac{GM_\odot}{q_c\left(1+e_c\right)}}\left[-\sin\theta_c\left(t_c-\tau\right)\vec{i}+\left[e_c+\cos\theta_c\left(t_c-\tau\right)\right]\vec{j}\,\right]. \tag{$C4$}$$

Along with equations (C1), (C3) and (C4), we manage to yield the final results for η, ξ as

$$\eta=r_c\left(t_c-\tau\right)\sin\psi\left(t_c-\tau\right)-\sqrt{\frac{GM_\odot}{q_c\left(1+e_c\right)}}\,\tau\left[\cos\psi(t_c-\tau)+e_c\cos\theta_c\left(t_c\right)\right], \tag{$C5.1$}$$

$$\xi=r_c\left(t_c-\tau\right)\cos\psi(t_c-\tau)-r_c\left(t_c\right)+\sqrt{\frac{GM_\odot}{q_c\left(1+e_c\right)}}\,\tau\left[\sin\psi(t_c-\tau)+e_c\sin\theta_c\left(t_c\right)\right], \tag{$C5.2$}$$

whence $\psi(t_c-\tau)=\theta_c(t_c)-\theta_c(t_c-\tau)$.

Eventually we have accomplished all of the discussions about yielding and solving the dust particles motions under every of the circumstance. In practical, we need to apply the equations accordingly.

REFERENCE

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