On Planetary Apparent Magnitude

(Preliminary)

I have been obsessed by the Kuiper Belt Objects (KBOs) since the day I got acquaintance with them, in that their distant and dim natural appearances allure and entice me to explore further. As these objects distribute afar from our Earth within the solar system, therefore, it is no easy to acquire the knowledge about how big they actually are by means of ordinary procedures such as conducting measurements about the apparent size through a telescope. However, it is reasonably easier to estimate their actual size in accordance with their brightness. Thereby I feel it necessary to explore the mechanisms, despite somewhat superficial due to my restricting knowledge, behind planetary apparent magnitude.

Here we mainly concentrate on the visual wavelengths from the planets and all other objects in the solar system without taking the thermal and radio wave radiation into consideration. All of these bodies only reflect the radiation of the Sun. The brightness of a body depends upon its distance from the Sun and the Earth, upon the size of its effective cross section that receives the radiation of the Sun, and upon the albedo, which measures the ability of an object to reflect light, of the surface.

I will try to exploit different perspectives to approximate the reality. First, Let us employ flux density F ($[F] = W \cdot m^{-2}$) to define the magnitude of a body. We standardize that the magnitude 0 corresponds to a specified flux density F_0. Hence, we have the following relationship:

$$m = -2.5\, lg \frac{F}{F_0} \tag{1}$$

If we denote the luminosity of the Sun is $L_\odot$, as the flux density is inversely proportional to the square of the distance, furthermore, we assume that the radiation is isotropic, according to the definition, the flux density at the distance r is:

$$F = \frac{L_\odot}{4\pi r^2} \tag{2}$$

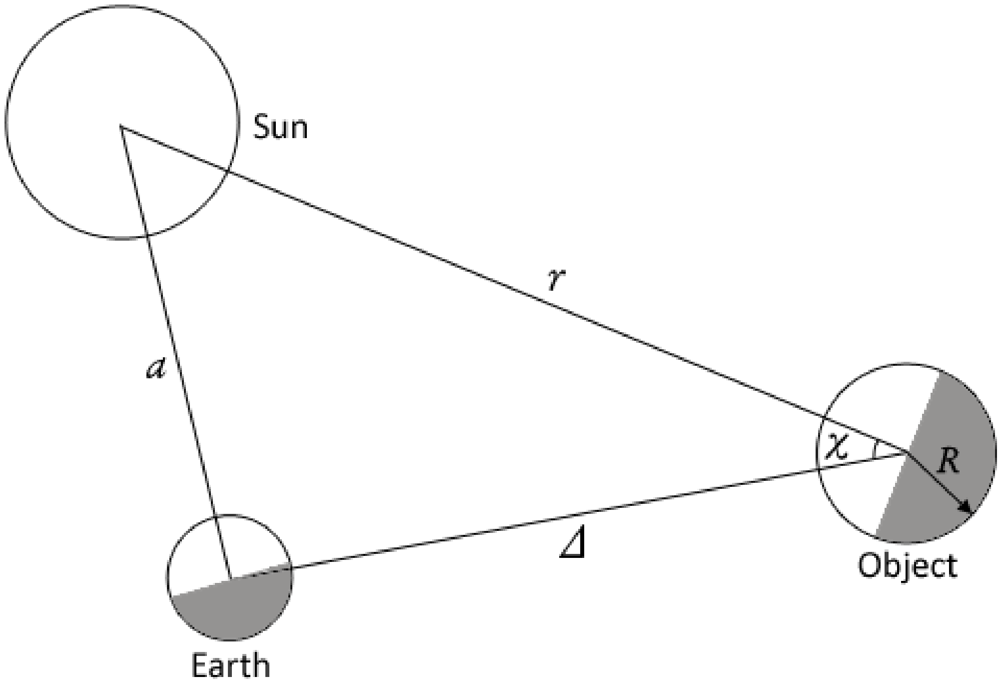

Fig-1. Symbols used in formulae.

If the radius of the planet is R, the area of its cross section is πR^2, and therefore the total flux incident on its surface should be:

$$L_i = \pi R^2 F = \pi R^2 \frac{L_\odot}{4\pi r^2} = \frac{L_\odot R^2}{4r^2} \tag{3}$$

Yet not all the total flux that the surface receives is reflected back. Instead, only a part of the original amount does while the rest is absorbed and converted into heat or other aspects. We now introduce a term called spherical albedo, or known as Bond albedo, denoted as A, to define the ability of reflection as a ratio of the emergent flux to the incident one. Obviously we have the restriction of $0 < A < 1$. Hence the flux reflected by the planet is:

$$L_r = AL_i = A\frac{L_\odot R^2}{4r^2} \tag{4}$$

Consider we observe the planet at a distance of $\varDelta$. If the radiation from the planet in visual wavelengths distributes isotropically, the flux density received on Earth should be:

$$F = \frac{L_r}{4\pi\varDelta^2} = A\frac{L_\odot R^2}{16\pi\varDelta^2 r^2} \tag{5}$$

Unfortunately this is not the case, as the radiation is anisotropically reflected. To simplify the scenario, we assume that the surface of the planet is homogeneous, and therefore the distribution of the radiation solely depends upon the phase angle, denoted as χ. We express the true flux density we receive is in the form

$$F = C\Phi(\chi)\frac{L_r}{4\pi\Delta^2} = C\Phi(\chi)\frac{AL_i}{4\pi\Delta^2} \tag{6}$$

where C is a constant while $\Phi(\chi)$ is called phase function. Specifically, we define that when $\chi = 0°$, $\Phi = 1$.
Rewrite the equation (6):

$$F = \frac{CA}{4\pi}\Phi(\chi)\frac{1}{\Delta^2}L_i = CA\Phi(\chi)\frac{L_\odot R^2}{16\pi\Delta^2 r^2} \tag{7}$$

We can draw a conclusion that the first factor intrinsically relates to the ability of reflection, denoted as β specifically for brevity, the second factor depends upon the phase angle, the third factor the distance dependence, and the fourth the incident flux.
Since all the reflected radiation should be found somewhere on some spherical surface, we must have

$$L_r = \int_S F\,dS = \int_S C\Phi(\chi)\frac{L_r}{4\pi\Delta^2}\,dS$$

or

$$\frac{4\pi\Delta^2}{C} = \int_S \Phi(\chi)\,dS \tag{8}$$

If the integration surface is the sphere with the radius Δ, as an infinitesimal on the surface can be described as $dS = \Delta^2 \sin\chi\, d\varphi d\chi$, we thus derive

$$\int_S \Phi(\chi)\,dS = \Delta^2\int_S \Phi(\chi)\sin\chi\,d\varphi d\chi = \Delta^2\int_{\varphi=0}^{2\pi}\int_{\chi=0}^{\pi}\Phi(\chi)\sin\chi\,d\varphi d\chi$$

$$= 2\pi\Delta^2\int_{\chi=0}^{\pi}\Phi(\chi)\sin\chi\,d\chi \tag{9}$$

Hence, the constant C should obey the following relationship:

$$C = \frac{4\pi\Delta^2}{\int_S \Phi(\chi)\,dS} = \frac{2}{\int_{\chi=0}^{\pi}\Phi(\chi)\sin\chi\,d\chi}$$

(10)

Now we try to find out the expression of Bond albedo A:

$$A=\frac{4\pi\beta}{C}=\frac{4\pi\beta}{\frac{2}{\int_{\chi=0}^{\pi}\Phi(\chi)\, sin\,\chi\, d\chi}}=2\pi\beta\int_{\chi=0}^{\pi}\Phi(\chi)\, sin\,\chi\, d\chi$$

(11)

We hereby introduce a Lambertian surface, namely, the surface is absolutely white and diffuse, it reflects all the radiation that receives, and moreover, its surface brightness remains totally the same for all viewing directions. From the definition, we know that the Bond albedo $A=1$ for a Lambertian surface and the phase function becomes:

$$\Phi(\chi)=\begin{cases} cos\,\chi\,, & \text{when } 0\le\chi<\frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

(12)

Therefore, by substituting the defined $\Phi(\chi)$, we can derive

$$C=\frac{2}{\int_{\chi=0}^{\frac{\pi}{2}} cos\,\chi\, sin\,\chi\, d\chi}=4$$

(13)

$$\beta=\frac{1}{\pi}$$

(14)

Hardly does any kind of an authentic Lambertian surface exist in reality. Yet it is a good approximation nevertheless.

Consider that the reflected flux density at phase angle $\chi=0°$, namely, $\Phi(\chi)=1$, is

$$F=\frac{CA}{4\pi}\frac{1}{\Delta^2}L_i$$

As for a Lambertian surface with the same size, we have:

$$F_L=\frac{4}{4\pi}\frac{1}{\Delta^2}L_i$$

Hence the ratio of the flux density of a non-Lambertian surface to the one of a Lambertian surface is

$$\frac{F}{F_L}=\frac{CA}{4}$$

(15)

Remember that we once have introduced a coefficient β to define the reflection ability. Now we in particularly introduce yet another one called geometric albedo, denoted as Γ, which satisfies the following two relationships

$$\Gamma = \frac{CA}{4} = \pi\beta \tag{16}$$

$$A = 2\Gamma \int_{\chi=0}^{\pi} \Phi(\chi) \sin\chi \, d\chi \tag{17}$$

We have discussed a Lambertian surface before. By implementing the relationship (16), we know that the geometric albedo of this kind of surface is $\Gamma = \pi\beta = 1$. Furthermore, we can interpret the physical meaning of the geometric albedo as the ratio of the flux density reflected by a planet to the one of a Lambertian surface with the same cross section at phase angle $\chi = 0°$.

DO NOT get confused by the equation (17), which seemingly expresses the Bond albedo as a dependence of the phase function $\Phi(\chi)$. Instead, it is the geometric albedo that is related to the phase function.

Next we move on to derive a formula for planetary magnitudes.

We receive the flux density from the Sun on Earth at a distance $a = 1$ AU is:

$$F_{\odot} = \frac{L_{\odot}}{4\pi a^2} \tag{16}$$

Therefore, the ratio of the flux density from a planet to the one from the Sun is:

$$\frac{F}{F_{\odot}} = \frac{CA}{4}\Phi(\chi)\frac{a^2R^2}{\Delta^2r^2} = \Gamma\Phi(\chi)\frac{a^2R^2}{\Delta^2r^2} \tag{17}$$

If the apparent magnitude of the Sun on Earth is $m_{\odot}$, the apparent magnitude of the planet denoted as m, in accordance with the equation (1), we have

$$m - m_{\odot} = -2.5\, lg\frac{F}{F_{\odot}} = -2.5\, lg\, \Gamma\Phi(\chi)\frac{a^2R^2}{\Delta^2r^2} \tag{18}$$

We suppose that the planet were observed at a distance of 1 AU both from the Earth and the Sun, at a phase angle $\chi = 0°$, in the case of which, the magnitude is denoted as H, specifically called the absolute magnitude. Thus we have

$$H = m_{\odot} - 2.5\, lg\, \Gamma\frac{R^2}{a^2} \tag{19}$$

Substituted with equation (19), equation (18) yields

$$m = H - 2.5\,lg\,\Phi(\chi)\frac{a^4}{\Delta^2 r^2} = H + 5\,lg\frac{\Delta r}{a^2} - 2.5\,lg\,\Phi(\chi)$$

(20)

We can learn that the expression comprises of three terms – intrinsic properties of the planet, the distance dependence, and the thorniest term, the phase function. By means of observations, we can acquire the apparent magnitude of the planet without too much difficulty. Moreover, attaining the specified distance will not be problematic either. Hence as time varies, the phase angle varies as well, and the phase function could be solved. However, after scrutiny we know that things will not become thus easy as we might expect. For inferior planets, namely, Mercury and Venus, as well as some small bodies that will come towards the Sun closer than the Earth does, the phase angle varies from 0° to 180°. Rather than directly acquire the phase function, we will obtain the sum of the absolute magnitude at phase angle $\chi = 0°$:

$$H(\chi) \triangleq H - 2.5\,lg\,\Phi(\chi)$$

(21)

which can be interpreted as the absolute magnitude at phase angle χ.

Yet for superior planets, the phase angle only covers a span of the whole. The greater distance it is at, the more limited the phase angle varies. Suppose a KBO that is circularly orbiting around the Sun at a distance 40 AU. The greatest phase angle is (based upon the assumption that the Earth is in a circular orbit around the Sun as well with an orbital radius 1 AU) at the time when the Earth appears at the greatest elongations seen from the object,

$$\chi_{max} = arcsin\frac{r}{a} = arcsin\frac{1}{40} \approx 1.4°$$

How small! A tiny lapse or error in observation may well lead to a sheer incorrect phase function at small angles, let alone to extrapolate!

Taken aback? We are thus trapped in a quagmire? By no means! Give a second thought, for those who show no much interest in analyzing the phase function but are interested in groping the properties of distant objects, it will be convenient to estimate the size of the body, or, if the size is known already previously by other means, the geometric albedo can be thus known as the phase function $\Phi(\chi) \approx 1$!

Henceforth for distant object, by approximation, equation (18) becomes:

$$m_0 - m_\odot = -2.5\,lg\,\Gamma\frac{a^2 R^2}{\Delta^2 r^2}$$

(22)

or, rearranging the terms yields for whoever interest in analysis of size:

$$R = \frac{\Delta r}{\sqrt{\Gamma a}} 10^{\frac{m_{\odot} - m_0}{5}}$$

(23)

or, for those interested in analysis of albedo:

$$\Gamma = \left(\frac{\Delta r}{Ra}\right)^2 10^{\frac{2\left(m_{\odot} - m_0\right)}{5}}$$

(24)

For vividly illustration, let's take an example of **Eris = 2003 UB_{313}**, the most massive known dwarf planet in the solar system, a member of the scattered disc objects (SDOs). The apparent magnitude in V band during 2005 Jan 5,6 and 7 was 18.83 ± 0.02, at a heliocentric distance about 97.5 AU (M. E. Brown, C. A. Trujillo, & D. L. Rabinowitz, 2005) and a geocentric distance about 96.9 AU. The apparent magnitude of the Sun in V band is -26.74. By taking all of the value into equation (23), and employing the error transfer function we obtain:

$$R = \frac{1087 \pm 50}{\sqrt{\Gamma}} \ km$$

For objects within the solar system, the geometric albedo is normally less than 1, which implicates that the radius of Eris will be no less than 1087±50 km. Compared to the value $R = 1300\,^{-100}_{+200}\,km$ with the Spitzer Space Telescope (J. Stansberry et al. 2007), my calculated result is pretty close! Despite the radius in good accuracy still lies in dispute, in particularly an occultation on Nov 6 2010 may suggest a smaller radius of Eris, it demonstrates that the method is successful as a rough estimate. Once the geometric albedo is known, the model should become accurate.

You may have noticed that even if an assumption of $\Gamma = 1$ obtains the value of the radius quite close to the reality already. It thus suggests that Eris has a high albedo surface, which is given as 0.86 ± 0.07 in Wikipedia. Without sound source, I boldly guess that this albedo refers to the geometric albedo as a result of difficulty in obtaining the Bond albedo.

I will take more samples and try to apply other methods to derive the planetary formula that fits the reality as close as possible in the near future.

References

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