A Simple Approach to Solve the Orbital Radius of Planet's Satellite

The approach to solve the problem occurred to me when I was sleeping some day in 2005. I am able to clearly recollect that I dreamed a stunning view through a giant telescope of Io with subtle patterns in its apparent disk, moving steadily in front the host, Jupiter, in transit, casting a dark shadow alongside on the planet.

It sparked me with great delight when I woke up – I was capable of calculating the orbital radius of all the Galilean Satellites in this way!

With simple knowledge of trigonometry, I successfully achieved to my target. In fact, I think everybody who masters trigonometry well will have capability of playing this game. There is no need applying tactics with calculus or other complicated knowledge. Instead, some elementary mathematics will be sufficient to set off.

Here I try to expatiate the way I find the solution.

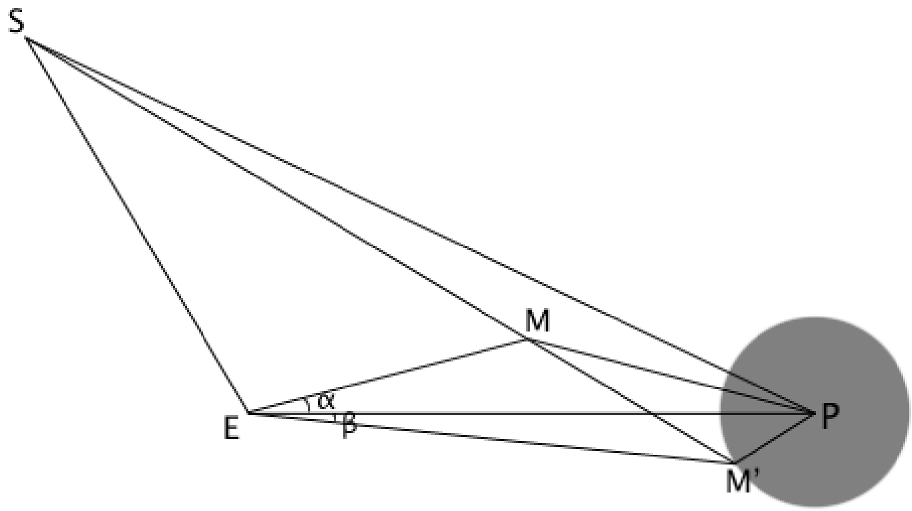

Fig-1

Let S be the Sun, E the earth, P the planet, and M the planet's satellite. The sun casts the satellite's shadow on the planet's surface at point M'. Observed from the earth, while the observer sees the shadow β in degree away from the planet's CM in the apparent disk, the satellite per se is α in degree away from the planet's CM in the other side. Here I define both of the angles are positive, and once the shadow and the satellite are both situated in the same side away from the CM, a negative correction with any of the angles will be valid.

To simplify the scenario, I set the assumption that the earth's radius is negligible in comparison to the geocentric distance of the planet and the heliocentric of the planet. In other words, the topocentric point is exactly the geocenter. Next, I presume that the satellite, the planetcenter, the heliocenter and the geocenter all share the same plane with each other. If the satellite deviates slightly outside the plane, we only need a tiny correction by taking a vertical displacement from the satellite perpendicularly to the plane into consideration. Furthermore, I need to emphasize that the following values should be know beforehand including the earth's and the planet's heliocentric distances, and normally the phase angle of the planet can be known, in the reality, without too much difficulty by means of observations, and thereby the geocentric distance of the planet can be solved as well.

I denote SE as r, SJ as r_p, EJ as $\mathit{\Delta}$, JM' as R, and JM as r_M. Note that the former four can be known in detailed value without too much difficulty, while the last one is exactly what we need to solve.

From the law of sine, we have the following equation with the triangle EJM':

$$\frac{\sin\beta}{R}=\frac{\sin\angle EJM'}{EM'}=\frac{\sin\angle EM'J}{\mathit{\Delta}} \qquad (1)$$

Hereby the triangle EJM' can be totally solved. By solving the equation (1), we yield that:

$$\angle EM'J = \arcsin\left(\frac{\mathit{\Delta}}{R}\sin\beta\right)$$

$$\angle EJM' = \pi - \beta - \arcsin\left(\frac{\mathit{\Delta}}{R}\sin\beta\right)$$

By implementing the law of cosines, we now are capable of solving EM' in a simpler way:

$$EM' = \sqrt{\mathit{\Delta}^2 + R^2 - 2\mathit{\Delta}R\cos\angle EJM'} = \sqrt{\mathit{\Delta}^2 + R^2 + 2\mathit{\Delta}R\cos\left[\beta + \arcsin\left(\frac{\mathit{\Delta}}{R}\sin\beta\right)\right]}$$

Henceforth we find that another triangle SEM' will be solved as well since two of its sides SE and EM' along with the angle $\angle SEM'$ between them can be all known. For brevity I denote the angle $\angle SEM'$ as Φ. Again both the law of sine and the law of cosines are employed:

$$\frac{\sin(\Phi+\beta)}{SM'}=\frac{\sin\angle SM'E}{r} \qquad (2)$$

$$SM' = \sqrt{r^2 + EM'^2 - 2EM'\cdot r\cos(\Phi+\beta)} \qquad (3)$$

With equation (2) and (3) we yield that

$$\angle MM^{'}E = \angle SM^{'}E = arcsin\frac{r\,sin(\Phi + \beta)}{\sqrt{r^2 + EM'^2 - 2EM' \cdot r\,cos(\Phi + \beta)}}$$

Hereby another triangle EMM' will be totally solved in a similar way. Prior to the destination, we need to work out the side EM of the triangle EMJ. We have:

$$\frac{sin\angle MM'E}{EM} = \frac{sin[\pi - (\alpha + \beta) - \angle MM'E]}{EM'} \qquad (4)$$

Or,

$$EM = \frac{sin\angle MM'E}{sin(\alpha + \beta + \angle MM'E)}EM^{'} \qquad (5)$$

As for the triangle EMJ, the law of cosines gives us:

$$r_M = JM = \sqrt{\Delta^2 + EM^2 - 2EM \cdot \Delta\,cos\,\alpha} \qquad (6)$$

where the EM can be calculated by equation (5).

Therefore, eventually, we arrive at the destination evidently that the value r_M is exactly what we need to solve.

Later I will use an example of Galilean Satellites and Jupiter to demonstrate the validity of my approach, and a comparison between the recognized value and the calculated one will be drawn.